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2002 J. Phys. A: Math. Gen. 35 1013

(http://iopscience.iop.org/0305-4470/35/4/314)

J. Phys. A: Math. Gen. **35** (2002) 1013–1024 PII: S0305-4470(02)27357-6

The lattice Toda field theory and lattice $\mathcal{W}$ algebras for B_2 and C_2

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Received 25 July 2001, in final form 9 November 2001
Published 18 January 2002
Online at stacks.iop.org/JPhysA/35/1013

Abstract
The lattice Toda field theory for finite-dimensional simple Lie algebras is studied. We show that the Poisson structure for the lattice Toda fields is closely related to that for the q-deformed $\mathcal{W}$ algebra. By making use of this relationship, we construct the lattice $\mathcal{W}$ algebra. We discuss the cases of B_2 and C_2 in detail, and associate them with the continuous theory.

PACS numbers: 02.30.Ik, 02.20.-a, 02.40.-k, 05.50.+q

1. Introduction

The extension and deformation of integrable nonlinear systems have been studied from many aspects of mathematical physics. Especially the Toda equation and its various extensions are paid much attention because of the rich mathematical structure [1–8]. We focus on an extension of the two-dimensional Toda equation based on simple Lie algebras [1], which is called the Toda field equation [5, 9],

$$\frac{\partial^2}{\partial t \partial x}\phi^{(i)} = -\sum_{j=1}^{l} B_{ij}\, \mathrm{e}^{\phi^{(j)}}. \tag{1.1}$$

Here $\phi^{(i)} \equiv \phi^{(i)}(x,t)$ ($i \in \{1, 2, \ldots, l\}$ and $t,\ x \in \mathbb{R}$) are Toda fields, and the space and time coordinates are denoted by x and t respectively. The matrix B_{ij} denotes a symmetrized Cartan matrix for the finite-dimensional simple Lie algebra $\mathfrak{g}$ whose rank is l. The Hamiltonian H for the Toda field equation (1.1) is given by

$$H = \sum_{i=1}^{l} Q_i \qquad Q_i = \int \mathrm{d}x\, \mathrm{e}^{\phi^{(i)}} \tag{1.2}$$

where Q_i are called the screening variables. It is well known that the Toda field theory realizes the $\mathcal{W}$ algebra [10, 11], and that the $\mathcal{W}$ algebra generates a kernel of the Hamiltonian (1.2) [5, 8].

The lattice Toda field theory describes a deformation of equation (1.1) to have a discrete space coordinate, while a time coordinate remains continuous. This was first introduced for $\mathfrak{g} = A_1$ as the lattice Liouville theory [12], and developed linking on the lattice $\mathcal{W}$ algebra for general A_l cases [13–18]. In [19], we introduced the lattice Toda field equation for general finite-dimensional simple Lie algebras $\mathfrak{g}$, whose Hamiltonian structure is described by the Poisson brackets on the lattice. Further we found that the lattice Toda fields are related to other integrable systems [19, 20]: the T system [21] and the q-deformed $\mathcal{W}$ algebra [22–24].

In this paper, as a sequel of our previous work [20] we study the correspondence of the Poisson structure for the lattice Toda fields and the q-deformed $\mathcal{W}$ algebra in detail, and apply it to construct the lattice $\mathcal{W}$ algebra. The Poisson structure for the q-deformed $\mathcal{W}$ algebra can be naturally translated into that on the lattice, since this q-deformed algebra is defined on the q-difference coordinate. Based on the study of the $\mathfrak{g} = A_l$ case [17, 25], in [20] we found that the Poisson brackets for one of the fundamental fields defined in the q-deformed theory are translated into those in the lattice Toda field theory for general $\mathfrak{g}$ cases. In this paper we further show that the correspondence reaches the crucial point to define the $\mathcal{W}$ algebra, namely the action of the screening operators on local fields in the q-deformed theory can be translated into the Poisson brackets of the local fields and the screening variables in the lattice theory. This fact allows us to construct the lattice $\mathcal{W}$ algebra by making use of the generators of the q-deformed $\mathcal{W}$ algebra. We concretely study the lattice $\mathcal{W}$ algebras for $\mathfrak{g} = B_2$ and C_2, which are the simplest examples with non-simply-laced Dynkin diagrams. It is remarkable that the lattice Miura transformation has a form which is not expected from the continuous theory [26]. We also discuss the continuous limit of the lattice $\mathcal{W}B_2, C_2$ currents.

This paper is arranged as follows. In section 2, after giving general formulae for the Lie algebras $\mathfrak{g}$, we introduce the lattice Toda field equation and its Hamiltonian structure following [19]. In section 3, we briefly review the q-deformed $\mathcal{W}$ algebra for $\mathfrak{g}$ [22]. In section 4, we discuss the correspondence of the lattice Toda field theory and the q-deformed $\mathcal{W}$ algebra. We give detailed results in the cases of classical simple Lie algebras. By using what is shown in section 4, we study the lattice $\mathcal{W}$ algebra for $\mathfrak{g} = B_2$ and C_2 in section 5. It is interesting to observe that the dual structure of B_2 and C_2 appears in the lattice $\mathcal{W}$ algebras. We also discuss the continuous limit of the lattice $\mathcal{W}$ currents and its symmetry. The last section, section 6, is devoted to the concluding remarks.

2. The lattice $\mathfrak{g}$-Toda field equation

2.1. General formulae for $\mathfrak{g}$

We give general formulae for Lie algebras, which are used in this paper. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank l, and we denote simple roots a_i where $i \in \mathcal{I} \equiv \{1, 2, \ldots, l\}$. The Cartan matrix $\boldsymbol{C} = (C_{ij})_{1 \leqslant i,j \leqslant l}$ is given by

$$C_{ij} = 2\,\frac{(a_i, a_j)}{(a_j, a_j)}$$

where (,) is an inner product. We define a diagonal matrix $\boldsymbol{D} = \mathrm{diag}(d_1, \ldots, d_l)$ with

$$d_i = \frac{(a_i, a_i)}{2}. \tag{2.1}$$

We also use the minimum value $d_{\min}$ (respectively the maximum value $d_{\max}$) of d_i; $d_{\min} = \min\{d_i \mid i \in \mathcal{I}\}$ ($d_{\max} = \max\{d_i \mid i \in \mathcal{I}\}$), and variables $d_{ij} = \min\{d_i, d_j\}$. The matrix $\boldsymbol{D}$ becomes an identity matrix when $\mathfrak{g}$ has a simply laced Dynkin diagram. In the non-simply-laced cases, $d_{\max}$ and $d_{\min}$ are given by $(d_{\max}, d_{\min}) = (1, \frac{1}{2})$ for $\mathfrak{g} = B_l, F_4$, $(2, 1)$ for C_l, and (3, 1) for G_2. We define two kinds of symmetrized Cartan matrix $\boldsymbol{B} = (B_{ij})_{i,j\in\mathcal{I}}$ and $\boldsymbol{\mathcal{B}} = (\mathcal{B}_{ij})_{i,j\in\mathcal{I}}$ by using the matrix $\boldsymbol{D}$ as

$$\begin{aligned} &\boldsymbol{B} = \boldsymbol{D}^{-1}\boldsymbol{C} \qquad B_{ij} = 4\,\frac{(a_i, a_j)}{(a_i, a_i)(a_j, a_j)} \\ &\boldsymbol{\mathcal{B}} = \boldsymbol{C}\,\boldsymbol{D} \qquad \mathcal{B}_{ij} = (a_i, a_j). \end{aligned} \tag{2.2}$$

We define an ordering of simple roots a_i following the Dynkin diagram where each vertex is related to a simple root in an ordinary way. For two simple roots a_i and a_j, we set $i \triangleleft j$ when the associated vertices are connected and $i < j$.

Further we let ω_i $(i \in \mathcal{I})$ be the fundamental weights which satisfy

$$(a_i, \omega_j) = \delta_{i,j}\frac{(a_i, a_i)}{2}.$$

For each ω_i, we denote by $R(\omega_i)$ the fundamental representation.

2.2. The lattice $\mathfrak{g}$-Toda field equations

Following [19], we introduce the lattice Toda field equation. Let $x_i(n)$ $(i \in \mathcal{I})$ be the lattice Toda fields defined on an infinite lattice $n \in d_{\min}\mathbb{Z}$, which satisfy the Poisson brackets

$$\begin{aligned} &\{x_i(n), x_i(m)\} = x_i(m)x_i(n), && \text{for} \quad n \equiv m (\mathrm{mod}\ d_i) \\ &\{x_i(n), x_j(m)\} = -\tfrac{1}{2}x_i(m)x_j(n) && \text{for} \quad i \triangleleft j \text{ or } j \triangleleft i \text{ and } n \equiv m\ (\mathrm{mod}\ d_{ij}) \\ &\{x_i(n), x_j(n)\} = -\tfrac{1}{2}x_i(n)x_j(n) && \text{for} \quad i \triangleleft j \end{aligned} \tag{2.3}$$

where we assume $n < m$. Others are Poisson commutative. For rational functions of $x_i(n)$, F and G, their Poisson bracket is given by

$$\{F,\ G\} = \sum_{i,j\in\mathcal{I},\ n,m\in d_{\min}\mathbb{Z}} \frac{\partial F}{\partial x_i(n)}\frac{\partial G}{\partial x_j(m)}\{x_i(n), x_j(m)\}.$$

Further we introduce the field $\beta_n^{(i)}$ $(i \in \mathcal{I},\ n \in d_{\min}\mathbb{Z})$ in terms of $x_i(n)$,

$$\beta_n^{(i)} = \frac{x_i(n)}{x_i(n + d_i)}. \tag{2.4}$$

The Poisson brackets for $\beta_n^{(i)}$ are obtained as

$$\begin{aligned} &\{\beta_n^{(i)}, \beta_m^{(i)}\} = (\delta_{m,n+d_i} - \delta_{m,n-d_i})\ \beta_n^{(i)}\beta_m^{(i)} && \text{for} \quad i \in \mathcal{I} \\ &\{\beta_n^{(i)}, \beta_m^{(j)}\} = (\delta_{m,n-d_j} - \delta_{m,n})\ \beta_n^{(i)}\beta_m^{(j)} && \text{for} \quad i \triangleleft j \text{ and } d_i \leqslant d_j \\ &\{\beta_n^{(i)}, \beta_m^{(j)}\} = (\delta_{m,n-d_j} - \delta_{m,n+d_j})\ \beta_n^{(i)}\beta_m^{(j)} && \text{for} \quad i \triangleleft j \text{ and } d_i > d_j. \end{aligned} \tag{2.5}$$

These Poisson brackets have an important feature, *locality*; a field $\beta_n^{(i)}$ does not Poisson commute only with the finite number of $\beta_m^{(j)}$, while the Poisson brackets (2.3) are *non-local*; a field $x_i(n)$ is not Poisson commutative with the infinite number of $x_j(m)$.

We describe the time evolution of the fields $\beta_n^{(i)}$ by $\frac{\partial}{\partial t}\beta_n^{(i)} = \{\mathcal{H}, \beta_n^{(i)}\}$, where the Hamiltonian $\mathcal{H}$ is composed of the screening variables on the lattice $\mathcal{Q}_i$;

$$\mathcal{H} = \sum_{i\in\mathcal{I}} \mathcal{Q}_i \qquad \mathcal{Q}_i = \sum_{n\in d_{\min}\mathbb{Z}} x_i(n). \tag{2.6}$$

Note that these are the lattice analogue of (1.2). Then we obtain a set of differential-difference equations that we call the lattice $\mathfrak{g}$-Toda field equations;

$$\frac{\partial}{\partial t}\log\beta_n^{(i)} = \left(\sum_{j\triangleleft i}\ \sum_{k=1}^{-C_{ij}} x_j(n + kd_{ij})\right) - x_i(n) - x_i(n + d_i) + \left(\sum_{i\triangleleft j}\ \sum_{k=0}^{-C_{ij}-1} x_j(n + kd_{ij})\right). \tag{2.7}$$

In the continuous limit of a coordinate n, fields $x_i(n)$ and $\log \beta_n^{(i)}$ reduce to $\exp(\phi^{(i)}(x))$ and $d_i \frac{\partial \phi^{(i)}(x)}{\partial x}$ respectively, and one sees that equation (2.7) becomes the $\mathfrak{g}$-Toda field equation (1.1). Remark that the matrix elements of $\boldsymbol{D}$ (2.1) appear in the continuous limit of $\beta_n^{(i)}$ because of the definition (2.4).

3. The q-deformed $\mathcal{W}$ algebra

We introduce the q-deformed $\mathcal{W}$ algebra for $\mathfrak{g}$ following [22, 23]. Note that we use slightly different notations in this paper. First we define the fields $A_i(z)$ and $Y_i(z)$ ($i \in \mathcal{I}, z \in \mathbb{C}$) associated with simple roots and fundamental weights of $\mathfrak{g}$ respectively. These fields satisfy the Poisson brackets;

$$\{A_i(z), A_j(w)\} = \mathcal{B}_{ij}\left(\frac{w}{z}\right) A_i(z)A_j(w) \tag{3.1}$$

$$\{Y_i(z), Y_j(w)\} = \mathcal{M}_{ij}\left(\frac{w}{z}\right) Y_i(z)Y_j(w) \tag{3.2}$$

where $\mathcal{B}_{ij}(z)$ and $\mathcal{M}_{ij}(z)$ are given by

$$\mathcal{B}_{ij}(z) = \sum_{m\in\mathbb{Z}} \mathcal{B}_{ij}^{q^m} z^m = \delta(q^{\mathcal{B}_{ij}} z) - \delta(q^{-\mathcal{B}_{ij}} z)$$

$$\mathcal{M}_{ij}(z) = \sum_{m\in\mathbb{Z}} \left(\boldsymbol{D}^{q^m} (\boldsymbol{\mathcal{B}}^{q^m})^{-1} \boldsymbol{D}^{q^m}\right)_{ij} z^m.$$

Here we set $\delta(z) = \sum_{m\in\mathbb{Z}} z^m$, and matrices $\boldsymbol{\mathcal{B}}^q = (\mathcal{B}_{ij}^q)_{i,j\in\mathcal{I}}$ and $\boldsymbol{D}^q = (\mathrm{D}_{ij}^q)_{i,j\in\mathcal{I}}$ are defined as

$$\mathcal{B}_{ij}^q = q^{\mathcal{B}_{ij}} - q^{-\mathcal{B}_{ij}} \qquad D_{ij}^q = \delta_{ij}(q^{d_i} - q^{-d_i})$$

with the symmetrized Cartan matrix $\boldsymbol{\mathcal{B}}$ (2.2). The fields $A_i(z)$ are written in terms of $Y_i(z)$ as follows [23]:

$$A_i(z) = \frac{Y_i(zq^{d_i})Y_i(zq^{-d_i})}{\prod_{j\neq i, C_{ij}=1} Y_j(z) \prod_{j\neq i, C_{ij}=2} Y_j(zq)Y_j(zq^{-1}) \prod_{j\neq i, C_{ij}=3} Y_j(zq^2)Y_j(z)Y_j(zq^{-2})}. \tag{3.3}$$

Next we introduce the screening operators $\boldsymbol{S}_i$ ($i \in \mathcal{I}$) which act on the fields $Y_i(z)$ as

$$\boldsymbol{S}_i \cdot Y_j(z)^{\pm} = \pm\delta_{i,j} Y_i(z)^{\pm} S_i(z) \tag{3.4}$$

and obey the Leibniz rule. To construct the q-deformed $\mathcal{W}$ algebra, we study the intersection of the kernel for these screening operators [22]. We consider the vector space spanned by

$$Y_1(zq^{m_1})^{k_1} \cdots Y_l(zq^{m_l})^{k_l} \qquad \text{for} \quad m_i, k_i \in \mathbb{Z} \tag{3.5}$$

where the action of $\boldsymbol{S}_i$ is naturally defined. We assign to each of these terms an element of the weight lattice $\sum_{i=1}^l k_i\omega_i$. It is shown that the maximal subspace which belongs to the kernel of the screening operator $\boldsymbol{S}_i$ exists, which is generated by the independent l fields $T_i(z)$ ($i \in \mathcal{I}$). The field $T_i(z)$ is related to the fundamental representation $R(\omega_i)$ and given by a sum of the terms (3.5), each of which is assigned to one of the weight vectors for $R(\omega_i)$. We let $\Lambda_j(z)$ ($j \in \mathcal{J}$, $\#\mathcal{J} = \dim R(w_1)$) denote a field which corresponds to the jth weight vector of $R(\omega_1)$ (see [22, 24] for the precise expression of $\Lambda_j(z)$ in terms of $A_i(z)$ and $Y_i(z)$). By definition, the field $T_1(z)$ is simply written as

$$T_1(z) = \sum_{j\in\mathcal{J}} \Lambda_j(z). \tag{3.6}$$

Note that not all the fields $\Lambda_j(z)$ are independent of each other. The fields $T_i(z)$ are proved to coincide with the q-deformed characters of the finite-dimensional representation of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ [23, 27].

By construction, the closed Poisson structure of the fields $T_i(z)$ defines the q-deformed $\mathcal{W}$ algebra. The variable transformation between $\Lambda_j(z)$ and $T_i(z)$ should correspond to the q-deformed Miura transformation where the fields $\Lambda_j(z)$ and $T_i(z)$ are regarded as the q-deformation of free fields and $\mathcal{W}$ currents respectively. It is natural to expect the q-deformed analogue of a Lax operator which gives the Miura transformation. To the best of our knowledge, such a Lax operator was discussed for $\mathfrak{g} = A_l$ [25] first, and was recently constructed for other classical simple Lie algebras [28]. We show the Lax operator for $\mathfrak{g} = B_2$ and C_2 in section 5, on the way to construct the lattice $\mathcal{W}$ algebra.

4. Correspondence of the q-deformation and the lattice

In this section we relate the q-deformed $\mathcal{W}$ algebra to the lattice Toda field theory in the classical $\mathfrak{g}$ cases. We introduced this relation in our previous work [20] by finding that the Poisson brackets for the fields $A_i(z)$ (3.1) are essentially same as that for the fields $\beta_n^{(i)}$ (2.5), and related the fields $\beta_n^{(i)}$ to the fields $\Lambda_j(z)$ (3.6). Besides the field $\Lambda_j(z)$, we now consider the field $Y_i(z)$ (3.2) to show that the action of the screening operator $\boldsymbol{S}_i$ (3.4) is identified with the Poisson structure on the lattice as $\boldsymbol{S}_i \cdot * \sim \{\mathcal{Q}_i, *\}$, where $\mathcal{Q}_i$ is the screening variable on the lattice (2.6).

First we define a map which translates the fields $\Lambda_j(z)$ into the lattice fields $\lambda_j(n)$ as [20]

$$\Lambda_j(zq^{2k}) \longmapsto \lambda_j(n - d_{\min} k) \qquad \text{for} \quad k \in \mathbb{Z} \tag{4.1}$$

where we assume $z = q^{-\frac{2}{d_{\min}} n}$. This map works on the fields $Y_i(z)$ in the cases of classical simple Lie algebras as follows:

$$\begin{aligned} Y_i(zq^{2k}) &\longmapsto y_i\left(n - d_{\min} k - \frac{i-1}{2}\right) \qquad \text{for} \quad i \in \mathcal{I} \setminus \{l\} \\ Y_l(zq^{2k}) &\longmapsto \begin{cases} y_l\left(n - k - \dfrac{l-1}{2}\right) & \text{for} \quad A_l \text{ and } C_l \\ y_l\left(n - \dfrac{k+l-2}{2}\right) & \text{for} \quad B_l \\ y_l\left(n - k - \dfrac{l-2}{2}\right) & \text{for} \quad D_l. \end{cases} \end{aligned} \tag{4.2}$$

Therefore the action of the screening operators $\boldsymbol{S}_i$ (3.4) on the lattice fields $y_i(n)$ becomes

$$\boldsymbol{S}_i \cdot y_j(n)^{\pm} = \pm\, \delta_{i,j}\; y_i(n)^{\pm} S_i(n). \tag{4.3}$$

Using the fields $\lambda_j(n)$ introduced above, next we write the fields $\beta_n^{(i)}$ in terms of $\lambda_i(n)$ as shown in [20]. Due to the transformation between $\Lambda_j(x)$ and $Y_i(z)$ [22, 30] and (4.2), we also obtain the $\beta_n^{(i)}$ in terms of $y_i(n)$. In the following we show these for the classical $\mathfrak{g}$ cases:

- A_l case

$$\beta_n^{(i)} = \frac{\lambda_{i+1}(n)}{\lambda_i(n)} = \frac{y_{i+1}(n) y_{i-1}(n+1)}{y_i(n) y_i(n+1)} \tag{4.4}$$

for $i \in \mathcal{I}$ and $y_0(n) = y_{l+1}(n) \equiv 1$.

- B_l case

$$\beta_n^{(i)} = \frac{\lambda_{i+1}(n)}{\lambda_i(n)} = \frac{\lambda_{\bar{i}}(n-l+i+\frac{1}{2})}{\lambda_{\overline{i+1}}(n-l+i+\frac{1}{2})} = \begin{cases} \dfrac{y_{i+1}(n)y_{i-1}(n+1)}{y_i(n)y_i(n+1)} & \text{for} \quad i \in \mathcal{I}\setminus\{l-1,l\} \\ \dfrac{y_l(n)y_l(n+\frac{1}{2})y_{l-2}(n+1)}{y_{l-1}(n+1)y_{l-1}(n)} & \text{for} \quad i = l-1 \end{cases}$$
$$\beta_n^{(l)} = \frac{\lambda_{\bar{l}}(n)}{\lambda_0(n)} = \frac{\lambda_0(n-\frac{1}{2})}{\lambda_l(n-\frac{1}{2})} = \frac{y_{l-1}(n+\frac{1}{2})}{y_l(n+\frac{1}{2})y_l(n)}. \tag{4.5}$$

- C_l case

$$\beta_n^{(i)} = \frac{\lambda_{i+1}(n)}{\lambda_i(n)} = \frac{\lambda_{\bar{i}}(n-l+i-1)}{\lambda_{\overline{i+1}}(n-l+i-1)} = \frac{y_{i+1}(n)y_{i-1}(n+1)}{y_i(n)y_i(n+1)} \qquad \text{for} \quad i \in \mathcal{I}\setminus\{l\}$$
$$\beta_n^{(l)} = \frac{\lambda_{\bar{l}}(n)}{\lambda_l(n)} = \frac{y_{l-1}(n+1)y_{l-1}(n+2)}{y_l(n+2)y_l(n)}. \tag{4.6}$$

- D_l case

$$\beta_n^{(i)} = \frac{\lambda_{i+1}(n)}{\lambda_i(n)} = \frac{\lambda_{\bar{i}}(n-l+i+1)}{\lambda_{\overline{i+1}}(n-l+i+1)} = \begin{cases} \dfrac{y_{i+1}(n)y_{i-1}(n+1)}{y_i(n)y_i(n+1)} & \text{for} \quad i \in \mathcal{I}\setminus\{l-2,l-1,l\} \\ \dfrac{y_l(n)y_{l-1}(n)y_{l-3}(n+1)}{y_{l-2}(n+1)y_{l-2}(n)} & \text{for} \quad i = l-2 \\ \dfrac{y_{l-2}(n+1)}{y_{l-1}(n+1)y_{l-1}(n)} & \text{for} \quad i = l-1 \end{cases}$$
$$\beta_n^{(l)} = \frac{\lambda_{\bar{l}}(n)}{\lambda_{l-1}(n)} = \frac{\lambda_{\overline{l-1}}(n)}{\lambda_l(n)} = \frac{y_{l-2}(n+1)}{y_l(n+1)y_l(n)}.$$

One now sees the correspondence of $A_i(z)$ and $\beta_n^{(i)}$ without referring to the Poisson brackets; the field $\beta_n^{(i)}$ written in terms of $y_i(n)$ can be identified with the fields $A_i(z)$ (3.3) by the map (4.2). For instance, in the $\mathfrak{g} = A_l$ case we obtain

$$A_i(zq^{-i}) \longmapsto \frac{y_i(n)y_i(n+1)}{y_{i+1}(n)y_{i-1}(n+1)}$$

which is equal to $(\beta_n^{(i)})^{-1}$. In general $\mathfrak{g}$ cases, we have checked that the correspondence of $A_i(z)$ and $\beta_n^{(i)}$ and a map

$$\delta\left(\frac{w}{z}q^{2k}\right) \longmapsto \delta_{m,n+d_{\min}k}$$

derive the Poisson brackets (2.5) from (3.1). Finally, by identifying $S_i(n)$ with $x_i(n)$, the direct calculations show that the action of the screening operator $\boldsymbol{S}_i$ on the fields $\beta_n^{(i')}$ coincides with the Poisson brackets of the screening variable $\mathcal{Q}_i$ (2.6) and $\beta_n^{(i')}$. In conclusion, the q-difference fields $T_i(z)$ which generate the kernel of the screening operators $\boldsymbol{S}_i$ are mapped to the lattice fields which are Poisson commutative with the screening variables $\mathcal{Q}_i$. We denote this map as $T_i(z) \longmapsto t_i(n)$.

5. Lattice $\mathcal{W}$ algebra

We define the lattice $\mathcal{W}$ algebra as a kernel of $\mathcal{Q}_i$, which is generated by the lattice $\mathcal{W}$ currents written solely in terms of the fields $\beta_n^{(i)}$ (2.4). This is realized by making use of the fields $t_i(n)$. For $\mathfrak{g} = A_l$, the lattice $\mathcal{W}$ algebra and its integrable structure have been studied [16–18]. In the following we introduce the lattice $\mathcal{W}$ algebra for $\mathfrak{g} = C_2$ [20] and newly construct the lattice $\mathcal{W}$ currents for B_2. We also study the continuous limit of the lattice $\mathcal{W}$ currents.

5.1. Lattice $\mathcal{W}C_2$ algebra

Following [20], we introduce the $\mathcal{W}C_2$ algebra. We have sets $\mathcal{I} = \{1, 2\}$ and $\mathcal{J} = \{1, 2, \overline{2}, \overline{1}\}$, and define all fields on the lattice $n \in \mathbb{Z}$. We have fields $t_i(n)$ $(i \in \mathcal{I})$ written in terms of the fields $\lambda_j(n)$ $(j \in \mathcal{J})$;

$$\begin{aligned} t_1(n) &= \lambda_1(n) + \lambda_2(n) + \lambda_{\overline{2}}(n) + \lambda_{\overline{1}}(n) \\ t_2(n) &= \lambda_1(n+1)(\lambda_2(n) + \lambda_{\overline{2}}(n) + \lambda_{\overline{1}}(n)) + (\lambda_2(n+1) + \lambda_{\overline{2}}(n+1))\lambda_{\overline{1}}(n). \end{aligned} \tag{5.1}$$

These fields $t_1(n)$ and $t_2(n)$ are related to the fundamental representation $R(\omega_1)$ and $R(\omega_2)$ respectively. The fields (4.1) $\lambda_j(n)$ $(j \in \mathcal{J})$ satisfy two constraints

$$\lambda_1(n+4)\lambda_2(n+3)\lambda_{\overline{2}}(n+1)\lambda_{\overline{1}}(n) = 1 \qquad \lambda_1(n+3)\lambda_{\overline{1}}(n) = 1 \tag{5.2}$$

and they are written in terms of $y_i(n)$ $(i \in \mathcal{I})$ as

$$\lambda_1(n) = y_1(n) \qquad \lambda_2(n) = \frac{y_2(n)}{y_1(n+1)} \qquad \lambda_{\overline{2}} = \frac{y_1(n+2)}{y_2(n+2)} \qquad \lambda_{\overline{1}} = \frac{1}{y_1(n+3)}. \tag{5.3}$$

We define the Lax operator which gives the transformation from $\lambda_j(n)$ to $t_i(n)$

$$\begin{aligned} \mathcal{L}_{C_2}(n) &= (D - y_1(n+2))\left(D - \frac{y_2(n+1)}{y_1(n+2)}\right)\left(D^2 - \frac{y_2(n)}{y_2(n+1)}\right) \\ &\quad \times \left(D - \frac{y_1(n)}{y_2(n)}\right)\left(D - \frac{1}{y_1(n)}\right) \\ &= (D - \lambda_1(n+2))(D - \lambda_2(n+1))(D^2 - \Omega(n)) \\ &\quad \times (D - \lambda_{\overline{2}}(n-2))(D - \lambda_{\overline{1}}(n-3)) \\ &= D^6 - t_1(n+2)D^5 + t_2(n+1)D^4 - t_2(n)D^2 + t_1(n)D - 1 \end{aligned} \tag{5.4}$$

where D is a shift operator; $D^k f(n) = f(n+k)D^k$. In (5.4), the second equality is due to (5.3), and $\Omega(n)$ is given by

$$\Omega(n) = \lambda_1(n+1)\lambda_2(n)\lambda_{\overline{2}}(n-1)\lambda_{\overline{1}}(n-2) = (\lambda_1(n+2)\lambda_2(n+1)\lambda_{\overline{2}}(n-2)\lambda_{\overline{1}}(n-3))^{-1}.$$

The Lax operator (5.4) was originally constructed for the q-deformed theory [28].

As shown in the q-deformed $\mathcal{W}$ algebra for C_2 [22], the fields $t_i(n)$ construct the closed but *nonlocal* Poisson brackets. To obtain the *local* Poisson brackets, we define $\mathcal{W}C_2$ currents $W_n^{(i)}$ $(i \in \mathcal{I})$ by using the fields $t_i(n)$ (5.1) as

$$W_n^{(1)} = \frac{1}{t_1(n)\,t_1(n+3)} \qquad W_n^{(2)} = \frac{t_2(n)}{t_1(n)\,t_1(n+1)}. \tag{5.5}$$

We rewrite these $W_n^{(i)}$ in terms of the fields $\beta_n^{(i)}$ (2.5). Due to (4.6), one sees

$$\beta_n^{(1)} = \frac{\lambda_2(n)}{\lambda_1(n)} = \frac{\lambda_{\overline{1}}(n-2)}{\lambda_{\overline{2}}(n-2)} \qquad \beta_n^{(2)} = \frac{\lambda_{\overline{2}}(n)}{\lambda_2(n)}$$

and the currents (5.5) are transformed into

$$\begin{aligned} W_n^{(1)} &= \frac{\alpha_n^{(3)}}{\left(\alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)} + 1\right)\left(\alpha_{n+3}^{(1)} + \alpha_{n+3}^{(2)} + \alpha_{n+3}^{(3)} + 1\right)} \\ W_n^{(2)} &= \frac{\alpha_n^{(3)}\left(\alpha_{n+1}^{(1)} + \alpha_{n+1}^{(2)}\right) + \alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)}}{\left(\alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)} + 1\right)\left(\alpha_{n+1}^{(1)} + \alpha_{n+1}^{(2)} + \alpha_{n+1}^{(3)} + 1\right)} \end{aligned} \tag{5.6}$$

where we introduce fields $\alpha_n^{(i)}$ $(i = 1, 2, 3)$

$$\alpha_n^{(1)} = \beta_n^{(1)} \qquad \alpha_n^{(2)} = \alpha_n^{(1)}\beta_n^{(2)} \qquad \alpha_n^{(3)} = \alpha_n^{(2)}\beta_{n+2}^{(1)}.$$

We note that these $\alpha_n^{(i)}$ satisfy the following Poisson brackets:

$$\begin{aligned} &\{\alpha_n^{(i)}, \alpha_m^{(i)}\} = (\delta_{m,n+1} - \delta_{m,n-1})\, \alpha_n^{(i)}\alpha_m^{(i)} \qquad \text{for} \quad i = 1, 2 \\ &\{\alpha_n^{(3)}, \alpha_m^{(3)}\} = (\delta_{m,n+3} - \delta_{m,n+2} + \delta_{m,n+1} - \delta_{m,n-1} + \delta_{m,n-2} - \delta_{m,n-3})\, \alpha_n^{(3)}\alpha_m^{(3)} \\ &\{\alpha_n^{(1)}, \alpha_m^{(2)}\} = (\delta_{m,n+1} - \delta_{m,n} - \delta_{m,n-1} + \delta_{m,n-2})\, \alpha_n^{(1)}\alpha_m^{(2)} \\ &\{\alpha_n^{(i)}, \alpha_m^{(3)}\} = (\delta_{m,n+1} - \delta_{m,n} + \delta_{m,n-2} - \delta_{m,n-3})\, \alpha_n^{(i)}\alpha_m^{(3)} \qquad \text{for} \quad i = 1, 2. \end{aligned}$$

We regard (5.6) as the lattice Miura transformation for $\mathfrak{g} = C_2$ in a sense of [12,29]. One can check that the currents $W_n^{(i)}$ (5.6) are Poisson commutative with the Hamiltonian (2.6) for the C_2-Toda field equation by direct calculations. The generating relations for the lattice $\mathcal{W}C_2$ algebra are enumerated in the appendix.

5.2. *Lattice $\mathcal{W}B_2$ algebra*

For $\mathfrak{g} = B_2$, we have sets $\mathcal{I} = \{1, 2\}$ and $\mathcal{J} = \{1, 2, 0, \bar{2}, \bar{1}\}$, and all fields are defined on the fractional lattice $n \in \mathbb{Z}/2$. In this case, we need not only the fields $\lambda_i(n)$ but also $y_i(n)$ to describe the fields $t_i(n)$ $(i \in \mathcal{I})$:

$$\begin{aligned} t_1(n) &= \lambda_1(n) + \lambda_2(n) + \lambda_0(n) + \lambda_{\bar{1}}(n) + \lambda_{\bar{2}}(n) \\ t_2(n) &= \frac{1}{y_2(n+1)} + \frac{y_2(n+\frac{1}{2})}{y_1(n+1)} + \frac{y_1(n)}{y_2(n)} + y_2(n-\tfrac{1}{2}) \\ &= y_2(n+\tfrac{1}{2})\left(\frac{\lambda_0(n)}{\lambda_1(n)}\lambda_{\bar{2}}(n-\tfrac{1}{2}) + \lambda_0(n-\tfrac{1}{2}) + \lambda_{\bar{2}}(n-\tfrac{1}{2}) + \lambda_{\bar{1}}(n-\tfrac{1}{2})\right) \end{aligned} \tag{5.7}$$

where

$$\lambda_1(n) = y_1(n) \qquad \lambda_2(n) = \frac{y_2(n)y_2(n+\frac{1}{2})}{y_1(n+1)} \qquad \lambda_0(n) = \frac{y_2(n)}{y_2(n+1)}$$

$$\lambda_{\bar{2}}(n) = \frac{y_1(n+\frac{1}{2})}{y_2(n+\frac{1}{2})y_2(n+1)} \qquad \lambda_{\bar{1}}(n) = \frac{1}{y_1(n+\frac{3}{2})}.$$

We define the Lax operator which generates the transformation between $y_i(n)$ and $t_i(n)$

$$\begin{aligned} \mathcal{L}_{B_2}(n) &= \left(D^{\frac{1}{2}} - y_2(n+\tfrac{1}{2})\right)\left(D^{\frac{1}{2}} - \frac{y_1(n+\frac{1}{2})}{y_2(n+\frac{1}{2})}\right)\left(D - \frac{y_1(n)}{y_1(n+\frac{1}{2})}\right) \\ &\quad \times \left(D^{\frac{1}{2}} - \frac{y_2(n-\frac{1}{2})}{y_1(n)}\right)\left(D^{\frac{1}{2}} - \frac{1}{y_2(n-\frac{1}{2})}\right) \\ &= D^3 - t_2(n+1)D^{\frac{5}{2}} + t_1(n+\tfrac{1}{2})D^2 - t_1(n)D + t_2(n)D^{\frac{1}{2}} - 1. \end{aligned} \tag{5.8}$$

This Lax operator is different from that introduced in [28].

We remark that the dual structure of Lie algebras C_2 and B_2 appears in the Lax operators $\mathcal{L}_{C_2}(n)$ (5.4) and $\mathcal{L}_{B_2}(n)$ (5.8). Note that the factors of $\mathcal{L}_{C_2}(n)$ are basically composed of the terms in $t_1(n)$ (5.1) related to $R(\omega_1)$ of C_2, while the factors in $\mathcal{L}_{B_2}(n)$ are given by the terms in $t_2(n)$ (5.7) associated with $R(\omega_2)$ for B_2.

We define the lattice $\mathcal{W}B_2$ currents $W_n^{(i)}$ $(i \in \mathcal{I})$ in terms of fields $t_i(n)$ (5.7) as

$$W_n^{(1)} = \frac{1}{t_1(n)\, t_1(n+\frac{3}{2})} \qquad W_n^{(2)} = \frac{t_2(n+\frac{1}{2})\, t_2(n+1)}{t_1(n)\, t_1(n+1)}. \tag{5.9}$$

By using (4.5), we relate the fields $\lambda_j(n)$ to $\beta_n^{(i)}$ $(i \in \mathcal{I})$

$$\beta_n^{(1)} = \frac{\lambda_2(n)}{\lambda_1(n)} = \frac{\lambda_{\bar{1}}(n-\frac{1}{2})}{\lambda_{\bar{2}}(n-\frac{1}{2})} \qquad \beta_n^{(2)} = \frac{\lambda_{\bar{2}}(n)}{\lambda_0(n)} = \frac{\lambda_0(n-\frac{1}{2})}{\lambda_2(n-\frac{1}{2})} = \frac{\lambda_1(n+\frac{1}{2})}{\lambda_1(n+1)\lambda_2(n)}$$

and rewrite the $\mathcal{W}$ currents (5.9) as

$$\begin{aligned} W_n^{(1)} &= \frac{\alpha_n^{(4)}}{\big(\alpha_n^{(1)}+\alpha_n^{(2)}+\alpha_n^{(3)}+\alpha_n^{(4)}+1\big)\big(\alpha_{n+\frac{3}{2}}^{(1)}+\alpha_{n+\frac{3}{2}}^{(2)}+\alpha_{n+\frac{3}{2}}^{(3)}+\alpha_{n+\frac{3}{2}}^{(4)}+1\big)} \\ W_n^{(2)} &= \frac{\big(\alpha_n^{(2)}+\alpha_n^{(3)}+\alpha_n^{(4)}+\alpha_n^{(3)}\alpha_{n+\frac{1}{2}}^{(2)}\big)\big(\alpha_{n+\frac{1}{2}}^{(2)}+\alpha_{n+\frac{1}{2}}^{(3)}+\alpha_{n+\frac{1}{2}}^{(4)}+\alpha_{n+\frac{1}{2}}^{(3)}\alpha_{n+1}^{(2)}\big)}{\big(\alpha_n^{(1)}+\alpha_n^{(2)}+\alpha_n^{(3)}+\alpha_n^{(4)}+1\big)\big(\alpha_{n+1}^{(1)}+\alpha_{n+1}^{(2)}+\alpha_{n+1}^{(3)}+\alpha_{n+1}^{(4)}+1\big)\alpha_{n+\frac{1}{2}}^{(3)}}. \end{aligned} \tag{5.10}$$

These denote the lattice Miura transformation for $\mathfrak{g} = B_2$. Here the fields $\alpha_n^{(i)}$ $(i \in \{1, 2, 3, 4\})$ are given by

$$\alpha_n^{(1)} = \beta_n^{(1)} \qquad \alpha_n^{(2)} = \alpha_n^{(1)}\beta_{n+\frac{1}{2}}^{(2)} \qquad \alpha_n^{(3)} = \alpha_n^{(2)}\beta_n^{(2)} \qquad \alpha_n^{(4)} = \alpha_n^{(3)}\beta_{n+\frac{1}{2}}^{(1)}$$

and they satisfy the Poisson brackets,

$$\begin{aligned} &\{\alpha_n^{(i)}, \alpha_m^{(i)}\} = (\delta_{m,n+1} - \delta_{m,n-1})\, \alpha_n^{(i)}\alpha_m^{(i)} \qquad \text{for} \quad i = 1, 3 \\ &\{\alpha_n^{(2)}, \alpha_m^{(2)}\} = (\delta_{m,n+\frac{1}{2}} - \delta_{m,n-\frac{1}{2}})\, \alpha_n^{(2)}\alpha_m^{(2)} \\ &\{\alpha_n^{(4)}, \alpha_m^{(4)}\} = (\delta_{m,n+\frac{3}{2}} + \delta_{m,n+1} - \delta_{m,n+\frac{1}{2}} + \delta_{m,n-\frac{1}{2}} - \delta_{m,n-1} - \delta_{m,n-\frac{3}{2}})\, \alpha_n^{(4)}\alpha_m^{(4)} \\ &\{\alpha_n^{(i)}, \alpha_m^{(i+1)}\} = (\delta_{m,n+1} - \delta_{m,n})\, \alpha_n^{(i)}\alpha_m^{(i+1)} \qquad \text{for} \quad i = 1, 2 \\ &\{\alpha_n^{(1)}, \alpha_m^{(3)}\} = (\delta_{m,n+1} - \delta_{m,n+\frac{1}{2}} - \delta_{m,n} + \delta_{m,n-\frac{1}{2}})\, \alpha_n^{(1)}\alpha_m^{(3)} \\ &\{\alpha_n^{(i)}, \alpha_m^{(4)}\} = (\delta_{m,n+1} - \delta_{m,n} + \delta_{m,n-\frac{1}{2}} - \delta_{m,n-\frac{3}{2}})\, \alpha_n^{(i)}\alpha_m^{(4)} \qquad \text{for} \quad i = 1, 2, 3. \end{aligned}$$

5.3. *Continuous limit of $\mathcal{W}B_2$, C_2 currents*

To discuss the continuum limit of the lattice $\mathcal{W}$ currents $W_n^{(i)}$ for B_2 and C_2, we associate the lattice free fields $\lambda_i(n)$ $(i \in \mathcal{I})$ with the free fields $r_i(x)$ on the continuous coordinate $x \in \mathbb{R}$ [17] as

$$\lambda_1(n) = \exp\left[\epsilon\, r_1(x-(n-1)\,\epsilon)\right] \qquad \lambda_2(n) = \exp\left[\epsilon\, r_2(x-n\,\epsilon)\right] \tag{5.11}$$

where ϵ is an infinitesimal parameter. The lattice $\mathcal{W}$ currents $W_n^{(i)}$ (5.6), (5.10) are appropriate to take the continuous limit using the free fields, since they can be written solely in terms of $\lambda_1(n)$ and $\lambda_2(n)$ via $\beta_n^{(i)}$ (4.5), (4.6). In the following we study the continuous limit of the $\mathcal{W}$ currents for each case.

- C_2 case
  After writing the currents $W_n^{(i)}$ (5.6) in terms of $\lambda_i(n)$ (5.11), we expand them by ϵ and obtain

$$\begin{aligned} W_{n-1}^{(1)} &\to \frac{1}{16} + \frac{\epsilon^2}{32} w_1(x) + \frac{\epsilon^3}{32}\big(-w_1(x)' + \tilde{w}(x)\big) + \mathrm{O}(\epsilon^4) \\ W_n^{(2)} - W_{n-1}^{(1)} &\to \frac{1}{4} + \frac{\epsilon^4}{64}\big(4w_2(x) - 4w_1(x)'' - w_1(x)^2\big) + \mathrm{O}(\epsilon^5). \end{aligned}$$

Here we introduce the fields;

$$
\begin{aligned}
w_1(x) &= -r_1^2 - r_2^2 - 4r_1' - 2r_2' \\
w_2(x) &= -4r_1''' - r_2''' - 3r_1r_1'' + 2r_1r_2'' - r_2r_2'' + 4r_1'r_2' - (r_2')^2 \\
&\quad + 2(r_1^2r_2' + r_1r_2r_2' + r_1'r_2^2) + r_1^2r_2^2 \\
\tilde{w}(x) &= r_1'r_2 + r_2r_2' + r_1'' + r_2''
\end{aligned}
\tag{5.12}
$$

where $r_i \equiv r_i(x)$ and $o' \equiv \frac{\partial o}{\partial x}$. Though an unexpected field $\tilde{w}(x)$ is included in (5.12), the relation between $w_i(x)$ and $r_i(x)$ (5.12) is nothing but the well known Miura transformation whose Lax operator is given by

$$
\begin{aligned}
\mathcal{L} &= (\partial + r_1)(\partial + r_2)\,\partial\,(\partial - r_2)(\partial - r_1) \\
&= \partial^5 + w_1\partial^3 + \tfrac{3}{2}w_1'\partial^2 + w_2\partial + \tfrac{1}{2}(w_2' - \tfrac{1}{2}w_1''')
\end{aligned}
\tag{5.13}
$$

where ∂ is a quasi-differential operator, $\partial f(x) \equiv f'(x) + f(x)\partial$. We note that to derive the Miura transformation (5.12) we should consider the continuous limit of the currents $W_n^{(i)}$ rather than the fields $t_i(n)$.

- B_2 case

As well as the C_2 case, using (5.11) we expand the lattice $\mathcal{W}B_2$ currents (5.10) by ϵ and obtain

$$
W_n^{(1)} \to \frac{1}{25} + \frac{2\epsilon^2}{125}w_1(x) + \frac{\epsilon^3}{125}(-w_1'(x) + \tilde{w}(x)) + \mathrm{O}(\epsilon^4)
$$

$$
W_n^{(2)} - 3(W_{n-\frac{1}{2}}^{(1)} + W_n^{(1)}) \to \frac{2}{5} + \frac{\epsilon^4}{500}\big(-9w_1''(x) + 20w_2(x) - 8w_1(x)^2\big) + \mathrm{O}(\epsilon^5)
$$

where

$$
\begin{aligned}
w_1(x) &= -3r_1' - r_2' - r_1^2 - r_2^2 \\
w_2(x) &= -r_1''' - r_1r_1'' + r_1r_2'' + r_1'r_2' + r_1^2r_2^2 + r_1'r_2^2 + r_2'(2r_1r_2 + r_1^2) \\
\tilde{w}(x) &= 2r_1r_2' + r_1'r_2 + r_2r_2' - r_1'' + 2r_2''.
\end{aligned}
\tag{5.14}
$$

The transformation between $w_i(x)$ and $r_i(x)$ is the Miura transformation generated by the Lax operator;

$$
\begin{aligned}
\mathcal{L} &= (\partial + r_1)(\partial + r_2)(\partial - r_2)(\partial - r_1) \\
&= \partial^4 + w_1\partial^2 + w_1'\partial + w_2.
\end{aligned}
\tag{5.15}
$$

The Lax operators (5.13) and (5.15) are known to give the integrable structure of the *continuous* Toda field equations for B_2 and C_2 respectively. In constructing the Lax operator (5.13) ((5.15)), the free field $r_1(x)$ is associated with a fundamental weight ω_1 and each factor is assigned to one of the weight vectors of $R(\omega_1)$ of B_2 (C_2). The fields $w_1(x)$ and $w_2(x)$ (5.12) ((5.14)) are the dynamical variables for the B_2(C_2)-KdV equations.

We find that the lattice Toda field theories for B_2, C_2 have the lattice Miura transformations which reduce to the opposite symmetry in the continuous limit, namely the lattice $\mathcal{W}C_2$ currents (5.6) ($\mathcal{W}B_2$ currents (5.10)) are related to the Lax operator of B_2 (5.13) (C_2 (5.15)). This is due to the fact that the lattice Toda field theory is based on the symmetrized Cartan matrix $\boldsymbol{B}$ (2.2). In the C_2 case, the lattice free fields $\lambda_1(n)$ and $\lambda_2(n)$ are related to the first two weight vectors of $R(\omega_1)$, $\omega_1 = a_1 + \frac{1}{2}a_2$ and $(\omega_1 - a_1) = \frac{1}{2}a_2$ respectively. In the continuous limit, the root vectors are rescaled by the diagonal matrix $\boldsymbol{D}$ (2.2) as $a_i \to d_i a_i$ (that is $(a_1, a_2) \to (a_1, 2a_2)$) which changes the previous weight vectors to $\omega_1 = a_1 + a_2$ and $(\omega_1 - a_1) = a_2$. The rescaled weights remind us of the construction of $R(\omega_1)$ of B_2. Similarly, the continuous limit of the lattice free fields for B_2 relates to $R(\omega_1)$ of C_2. Recall that at the end of section 2 we have mentioned the continuous limit of the lattice Toda field equation and a field $\beta_n^{(i)}$ reduces to $d_i \frac{\partial \phi^{(i)}(x)}{\partial x}$.

6. Concluding remarks

In this paper, we have studied the lattice Toda field theory for finite-dimensional simple Lie algebras and applied it to construct the lattice $\mathcal{W}$ currents. Our main claims are as follows: (1) the Poisson structure for the lattice Toda fields can be identified with that for the q-deformed $\mathcal{W}$ algebra, and also describes the action of the q-deformed screening operators; (2) the correspondence (1) gives a clue to construct the lattice $\mathcal{W}$ algebra. We have shown (1) in the cases of classical simple Lie algebras, and concretely constructed the lattice $\mathcal{W}$ algebra for B_2 and C_2. The lattice $\mathcal{W}B_2, C_2$ algebras are the simplest examples associated with non-simply-laced Dynkin diagrams, and one sees that the dual structure of Lie algebras B_2 and C_2 appears in the lattice system. Because of the symmetrized Cartan matrix, the lattice Miura transformation for $\mathcal{W}C_2$ (B_2) currents is related to the continuous system of the opposite symmetry, the B_2 (C_2) KdV equation.

As a future problem, it is interesting to study the lattice $\mathfrak{g}$-Toda field theory based on the exchange algebra on the lattice, besides known A_l cases [31, 32]. To look for the discrete soliton equations related to the lattice $\mathcal{W}B_2, C_2$ algebras is also an open problem. These soliton equations may link to the extension of the Painlevé equation based on affine Weyl groups [33, 34], as inferred from the A_l case.

Acknowledgments

The author thanks Professor Atsuo Kuniba for fruitful discussions. She is grateful to Professor B de Wit and members of ITP and the Spinoza Institute at Utrecht University for their hospitality. She also thanks the referees for valuable advice which has improved this manuscript. RI is a Research Fellow of the Japanese Society for the Promotion of Science.

Appendix. The lattice $\mathcal{W}C_2$ algebra

After tedious calculations, we find that the currents (5.6) generate a closed algebra given by the following Poisson relations:

$$\begin{aligned}
&\{W_n^{(1)}, W_{n+1}^{(1)}\} = W_n^{(1)} W_{n+1}^{(1)} \big(1 - W_n^{(2)} - W_{n+3}^{(2)}\big) \\
&\{W_n^{(1)}, W_{n+2}^{(1)}\} = W_n^{(1)} W_{n+2}^{(1)} \big(-1 + W_{n+2}^{(2)}\big) \\
&\{W_n^{(1)}, W_{n+3}^{(1)}\} = W_n^{(1)} W_{n+3}^{(1)} \big(1 - W_n^{(1)} - W_{n+3}^{(1)}\big) \\
&\{W_n^{(1)}, W_{n+4}^{(1)}\} = -W_n^{(1)} W_{n+4}^{(1)} W_{n+3}^{(2)} \\
&\{W_n^{(1)}, W_{n+6}^{(1)}\} = -W_n^{(1)} W_{n+6}^{(1)} W_{n+3}^{(1)}
\end{aligned}$$

$$\begin{aligned}
&\{W_n^{(2)}, W_{n+4}^{(1)}\} = -W_n^{(2)} W_{n+4}^{(1)} W_{n+1}^{(1)} \\
&\{W_n^{(2)}, W_{n+3}^{(1)}\} = W_n^{(1)} W_{n+3}^{(1)} \big(1 - W_n^{(2)}\big) \\
&\{W_n^{(2)}, W_{n+2}^{(1)}\} = -W_n^{(2)} W_{n+2}^{(1)} W_{n+1}^{(2)} \\
&\{W_n^{(2)}, W_{n+1}^{(1)}\} = W_n^{(2)} W_{n+1}^{(1)} \big(1 - W_n^{(2)} - W_{n+1}^{(1)}\big) \\
&\{W_n^{(2)}, W_n^{(1)}\} = W_n^{(1)} \big(W_n^{(1)} - W_n^{(2)}\big)\big(1 - W_n^{(2)}\big) \\
&\{W_n^{(2)}, W_{n-1}^{(1)}\} = W_n^{(2)} W_{n-1}^{(1)} \big(W_{n-1}^{(2)} - W_{n+1}^{(2)}\big) \\
&\{W_n^{(2)}, W_{n-2}^{(1)}\} = -W_{n-2}^{(1)} \big(W_{n-2}^{(1)} - W_n^{(2)}\big)\big(1 - W_n^{(2)}\big)
\end{aligned}$$

$$\{W_n^{(2)}, W_{n-3}^{(1)}\} = W_n^{(2)} W_{n-3}^{(1)} \big(-1 + W_n^{(2)} + W_{n-3}^{(1)} \big)$$
$$\{W_n^{(2)}, W_{n-4}^{(1)}\} = W_n^{(2)} W_{n-4}^{(1)} W_{n-1}^{(2)}$$
$$\{W_n^{(2)}, W_{n-5}^{(1)}\} = W_{n-5}^{(1)} W_{n-2}^{(1)} \big(W_n^{(2)} - 1 \big)$$
$$\{W_n^{(2)}, W_{n-6}^{(1)}\} = W_n^{(2)} W_{n-6}^{(1)} W_{n-3}^{(1)}$$

$$\{W_n^{(2)}, W_{n+1}^{(2)}\} = W_n^{(2)} W_{n+1}^{(2)} \big(1 - W_n^{(2)} - W_{n+1}^{(2)} \big)$$
$$\{W_n^{(2)}, W_{n+2}^{(2)}\} = -W_n^{(2)} W_{n+2}^{(2)} \big(W_n^{(1)} + W_{n+1}^{(2)} \big) + W_n^{(1)} \big(W_n^{(2)} + W_{n+1}^{(2)} + W_{n+2}^{(2)} - 1 \big)$$
$$\{W_n^{(2)}, W_{n+3}^{(2)}\} = -W_n^{(2)} W_{n+3}^{(2)} \big(W_n^{(1)} + W_{n+1}^{(1)} \big) - W_{n+1}^{(1)} \big(W_n^{(1)} + W_n^{(2)} \big) + W_n^{(1)} W_{n+3}^{(2)}$$
$$\{W_n^{(2)}, W_{n+4}^{(2)}\} = -W_n^{(2)} W_{n+4}^{(2)} W_{n+1}^{(1)}.$$

Other Poisson brackets are obtained from the above or zero.

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